Some new results on consecutive equidivisible integers

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Abstract

We have found maximum possible runs of consecutive positive integers each having exactly k divisors for some fixed values of k. In addition, we exhibit the run of 10 consecutive positive integers each having exactly 12 divisors and two runs of 14 consecutive positive integers each having exactly 24 divisors.

1 Introduction

We let $\tau(n)$ denote the number of positive divisors of a positive integer n. Following [1], we say positive integers m and n to be *equidivisible* if $\tau(m) = \tau(n)$. For each $k \in \mathbb{Z}^+$, let $D(k)$ be the set of positive integers which begin maximal runs of equidivisible numbers with exactly k divisors. More precisely,

$$D(1) = \{1\} \quad \text{and} \quad D(k) = \{a \in \mathbb{Z}^+ \mid \tau(a) = k, \tau(a-1) \neq k\}, \quad k > 1.$$

We let $D(k, s)$ denote the set of numbers $a \in D(k)$ such that $\tau(a + i) = k$ for $0 \leq i \leq s - 1$ and $\tau(a + s) \neq k$. In other words, $D(k, s)$ is the set of positive integers n that start runs of s consecutive integers with exactly k divisors. For example, 33 and 85 both belong to $D(4, 3)$. Clearly, for a fixed k and varying s, the sets $D(k, s)$ are pairwise disjoint and form a partition of $D(k)$. For $n \in D(k)$ we write $L(n) = s$ if $n \in D(k, s)$.

It is easy to see that for each k, we have $D(k, s) = \varnothing$ for all large enough s. On the other hand, Erdös conjectured that for every s, $D(k, s) \neq \varnothing$ for some k, i.e., there exists sequences of consecutive equidivisible integers of any length (see [2, Problem B18]).

Let

$$M(k) = \max\{s \in \mathbb{Z}^+ \mid D(k, s) \neq \varnothing\}.$$

So $M(k) = 1$ for all odd k because two squares cannot be consecutive positive integers and $M(2) = 2$ because $(2, 3)$ is the only pair of consecutive primes. Another trivial example is $M(4) = 3$. Indeed, $L(33) = 3$ and if $n > 8$ is divisible by 4 then $\tau(n) \geq 4$.

Table 1 gives known ranges for $M(k)$ for every even $k < 30$, according to [1].

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k	2	4	6	8	10	12	14	16	18	20	22	24	26	28
M(k)	2	3	5	7	3	5-23	3	7	3-5	3-7	1-3	5-31	1-3	2-7

Table 1: Possible ranges of $M(k)$ for even $k < 30$ according to [1].

In 2006, Ali and Mishutka [3] showed that $M(24) \geq 12$.

We have found exact values of $M(k)$ for 55 fixed values of k. Also we have shown that $M(12) \geq 10$. Finally, we have found six runs of 13 and two runs of 14 consecutive equidivisible numbers each having exactly 24 divisors.

2 Results

2.1 Consecutive integers with $2p$ divisors

First of all, let us consider the case $k = 2p$, where $p > 3$ is prime. Düntsch and Eggleton [1] proved that $M(2p) \leq 3$. It seems that it is exactly equal to 3. Letting $n = 3^{p-1}r_1$, $n+1 = 2^{p-1}r_2$, $n+2 = q^{p-1}r_3$ or $n = q^{p-1}r_1$, $n+1 = 2^{p-1}r_2$, $n+2 = 3^{p-1}r_3$ for suitable primes q, r_1, r_2, r_3, we have found many examples of n in $D(2p,3)$ by Chinese remainder theorem. Moreover, for all these p we have found the smallest corresponding n. For instance, let $p = 11$. Then $n = 3^{10} \cdot 1765118938727$ starts the least triple for $q = 5$. Since $53^{10} > n$ it is sufficient to look over the triples for $q \leq 47$. It remains to note that for each of these q we need to check only a small number of cases.

p	n
5	7939375
7	3388031791
11	104228508212890623
13	1489106237081787109375
17	273062471666259918212890623
19	804505911103256259918212890623
23	490685203356467392256259918212890623
29	6794675247932944436619977392256259918212890623
31	329757106427071213106619977392256259918212890623
37	4459248710164424946384890995893380022607743740081787109375
41	3685099958690838758895720896109004106619977392256259918212890623
43	1038001791494840815734697769103890995893380022607743740081787109375
47	12229485870130123102579152313423230896109004106619977392256259918212 890623

Table 2: Smallest elements n of $D(2p,3))$, for prime $5 \leq p \leq 47$.

The correspondence between p and n for all prime $5 \le p \le 47$ is given by the Table 2. In fact, we have found the necessary triples for every prime $p < 200$. Full list one can see at the website of Mathematical Marathon [4].

It was observed that $q = 5$ in all smallest triples we found. Of course, this property for smallest triples is not guaranteed for all p. While we have not yet observed a minimal triple with $q \ne 5$. The following example may serve as indirect evidence for their existence. For $p = 5$, $q = 19$, the smallest triple starts at $n = 130,358,662,767$, while $p = 5$, $q = 43$ give even a smaller start of the smallest triple at $n = 3,388,031,791$.

2.2 Exact values of $M(k)$ for k divisible by 4 and nondivisible by 3

Let k is a positive integer such that $k > 4$, $4 \mid k$ and $3 \nmid k$. If $n \equiv 4 \pmod 8$ then $3 \mid \tau(n)$. Hence it can not be equal to k. Therefore $M(k) \le 7$ and for every positive integer n starting a run of 7 numbers each having k divisors we have $n \equiv 5 \pmod 8$.

Apparently, for the considered values k $M(k)$ is exactly equal to 7.

We show the method of constructing of required runs for $k = 20$. Chinese remainder theorem provides existence of infinite number of positive integers n such that

$$\begin{aligned} n &= 3 \cdot 13^4 \cdot q_1, \\ n+1 &= 2 \cdot 17^4 \cdot q_2, \\ n+2 &= 7^4 \cdot q_3, \\ n+3 &= 3 \cdot 2^4 \cdot q_4, \\ n+4 &= 5^4 \cdot q_5, \\ n+5 &= 2 \cdot 11^4 \cdot q_6, \\ n+6 &= 3^4 \cdot q_7, \end{aligned} \tag{1}$$

and all prime factors of numbers $q_1, q_2, \ldots, q_7$ are greater than 17.

For positive integer $n = 76,043,484,008,534,356,379,398,200,621$ satisfying conditions (1) numbers q_1, q_2, q_4, q_6 are prime and $\tau(q_3) = \tau(q_5) = \tau(q_7) = 4$. Hence this n starts a run of 7 numbers each having 20 divisors.

Of course, conditions (1) are not necessary. Our choose of moduli is largely arbitrary.

We used a similar technique to obtain $M(k) = 7$ for some other values k. The correspondence between these k and n with $L(n) = M(k) = 7$ is given by the Table 3.

2.3 Exact value of $M(18)$

We describe the case of $k = 18$ in more detail. First of all, note that $\tau(n) = 18$ implies that n has one of the following forms: p^{17}, $p^8 q$, $p^5 q^2$, or $p^2 q^2 r$, where p, q, r are primes. It immediately follows that if prime $p \mid n$ but $p^2 \nmid n$ then $n = pa^2$ for some positive integer a coprime to p.

Let n starts a run of 5 consecutive numbers with 18 divisors each. Düntsch and Eggleton [1] showed that $n \equiv 1 \pmod 8$. Hence $n + 1 = 2a^2$ and $n + 3 = 4c$ where $a = pq$ or $a = p^4$, $c = r^2 s$ or $c = r^5$ for some odd primes p, q, r, s. Now we can prove that $n + 2$ is divisible by 3.

k	n
8	171893
16	17476613
20	76043484008534356379398200621
28	452785996182923067361779632166688093890621
32	788892193463818869
40	1469311698340824775996499340503749
44	549796909842455469360784994463197630002088937189442381988212890621
52	1168449695271445704188430860163238225121312942349053407088447908229 18212890621
56	2255380110035375413875832363238406970059584 3749
64	7828102675311442968 69
80	30182284844951863821363057928337 33749
100	2850722870023179358438958811488353717139285841472441554229789510676250621
112	10289638188241284772556486723898957511531346384 3749
128	59157122333917084370843503998 69

Table 3: Elements n of $D(k, 7))$ for which $L(n) = M(k)$.

We have 3 possibilities: $3 \mid n$, $3 \mid (n+1)$, or $3 \mid (n+2)$.

If $3 \mid n$ then $n = 3b^2$ or $n = 9b$. In the former case, we have $3b^2 + 1 = 2a^2$, which has no solutions modulo 3. In the latter case, we have $n + 3 = 12p^2$. Considering $n + 1 = 2a^2$ one can get $2a^2 + 2 = 12p^2$, which again has no solutions modulo 3. Therefore n is not divisible by 3.

If $3 \mid (n+1)$ then $n+1 = 18p^2$ for some prime $p > 3$ (the case of $n+1 = 2 \cdot 3^8$ is obviously impossible). Hence $n + 4 = 3b^2$, implying that $6p^2 = b^2 - 1$, which is impossible since the right hand side is divisible by 8 and $6p^2 \equiv 6 \pmod 8$. Therefore $n + 1$ is not divisible by 3 either.

Hence, we must have $3 \mid (n + 2)$, which we satisfy by assuming that $n + 2 = 9b$. Since one of the five consecutive numbers must be divisible by 5, we let $n + 4 = 25d$. These assumptions fulfill the divisibility requirements by small primes.

We further find it convenient to assume the following forms of the integers: $n = p_1^2 p_2^2 p_3$, $n+2 = 9p_4^2 p_5$, $n+3 = 4p_6^2 p_7$, $n+4 = 25p_8^2 p_9$, where $p_1, p_2, \ldots, p_9$ are distinct primes greater than 5.

Let $m_1 = p_1^2 p_2^2$, $m_2 = 9p_4^2$, $m_3 = 4p_6^2$, and $m_4 = 25p_8^2$. We choose p_1, p_2, p_4, p_6, and p_8 be small primes such that the following system of quadratic congruences is solvable:

$$\begin{cases} 2x^2 - 1 \equiv 0 \pmod{m_1}, \\ 2x^2 + 1 \equiv 0 \pmod{m_2}, \\ 2x^2 + 2 \equiv 0 \pmod{m_3}, \\ 2x^2 + 3 \equiv 0 \pmod{m_4} \end{cases} \tag{2}$$

Since the moduli in system (2) are pairwise coprime and each congruence has 4 solutions, by Chinese remainder theorem the system solution consists of 256 classes of positive integers $\{x_i + jm\}_{j \geq 0}$, where $1 \leq i \leq 256$ and $0 \leq x_i < m = m_1 m_2 m_3 m_4$. Each class can give us $n = 2(x_i + jm)^2 - 1$ as soon as n/m_1, $(n+2)/m_2$, $(n+3)/m_3$, $(n+4)/m_4$ are prime and $x_i + jm = qr$ or $x_i + jm = q^4$ for some primes q, r.

The smallest known n starting 5 numbers with 18 divisors each was obtained for $p_1 = 7$, $p_2 = 17$, $p_4 = 11$, $p_6 = 13$, $p_8 = 29$, and $j = 260$. It is equal to

$$6,481,049,360,854,613,144,556,866,375,483,521.$$

Using computer search one can obtain many other numbers (for the aforementioned and other values of m_1, m_2, m_3, m_4) in $D(18, 5)$.

2.4 New lower bounds for $M(12)$ and $M(24)$

We have found n with $\tau(n) = 12$ and $L(n) = 10$, implying that $M(12) \geq 10$.

We have searched desired run looking over numbers n such that

$$\begin{aligned} &n = 19^2 \cdot q_1; \\ &n+1 = 2 \cdot 5^2 \cdot q_2; \\ &n+2 = 3 \cdot 17^2 \cdot q_3; \\ &n+3 = 2^2 \cdot q_4; \\ &n+4 = 23^2 \cdot q_5; \\ &n+5 = 2 \cdot 3^2 \cdot q_6; \\ &n+6 = 5 \cdot 7^2 \cdot p; \\ &n+7 = 2^5 \cdot q_7; \\ &n+8 = 3 \cdot 13^2 \cdot q_8; \\ &n+9 = 2 \cdot 11^2 \cdot q_9. \end{aligned} \tag{3}$$

First of all, these conditions fulfill the divisibility requirements by 2, 3, 5, 7 and 11. We choose additional conditions to increase empiric probability of required number of divisors for psitive integers q_i.

Let $a = 5 \cdot 7^2$. Using Chinese remainder theorem we have found the least positive integer

$p_0 = 7,623,414,751,537,859$, satisfying to the system of linear congruences:

$$\begin{cases} ax^2 - 6 \equiv 0 \pmod{19^2}, \\ ax^2 - 5 \equiv 0 \pmod{5^2}, \\ ax^2 - 4 \equiv 0 \pmod{17^2}, \\ ax^2 - 2 \equiv 0 \pmod{23^2}, \\ ax^2 - 1 \equiv 0 \pmod{3^2}, \\ ax^2 + 1 \equiv 0 \pmod{2^5}, \\ ax^2 + 2 \equiv 0 \pmod{13^2}, \\ ax^2 + 3 \equiv 0 \pmod{11^2} \end{cases}$$

We have looked over positive integers $p = p_0 + jm$, where $m = 2^6 \cdot 3^3 \cdot 5 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23^2$. For each p which is prime we can obtain $n = ap - 6$ and numbers $q_1, q_2, \ldots, q_9$ from conditions (3). It is enough to us to find j for which $q_2, q_3, q_6, q_7, q_8, q_9$ are prime and $\tau(q_1) = \tau(q_4) = \tau(q_5) = 4$. The smallest j satisfying these conditions is $j = 647,045,875$, which corresponds to $n = 1,545,780,028,345,667,311,380,575,449$ starting a run of 10 consecutive equidivisible numbers each having exactly 12 divisors.

Using similar construction, we have also found six runs of 13 consecutive numbers each having exactly 24 divisors. Smallest of them starts at

$$58,032,555,961,853,414,629,544,105,797,569.$$

This number gives the upper bound for A006558(13) in the OEIS [5] (A006558(n) starts the first run of n consecutive integers with same number of divisors).

Finally, we have found two runs of 14 consecutive equidivisible numbers each having exactly 24 divisors. The first run starts at

$$25,335,305,376,270,095,455,498,383,578,391,968,$$

which was obtained in the form $3 \cdot 11 \cdot 23^2(p_0 + jm) - 13$ with $m = 331,805,549,004,454,324,800$, $p_0 = 92,513,784,488,630,385,533$, and $j = 4,373,940,659$. The second run starts at

$$54,546,232,085,777,926,508,945,202,650,399,569,568,$$

which was obtained for the same values of m and p_0 but $j = 9,416,976,775,575$. Thus $M(24) \geq 14$. Of course, the smallest of these numbers gives the upper bound for A006558(14).

Our results confirm the Erdös conjecture for $s \leq 14$.

2.5 Smallest elements of $D(k, s)$

Table 4 gives a smallest known element $n \in D(k, s)$ for every even k below 30, where s is greatest known number with $D(k, s) \neq \varnothing$.

k	n	$L(n)$	$M(k)$
2	**2**	2	2
4	**33**	3	3
6	**10093613546512321**	5	5
8	**171893**	7	7
10	**7939375**	3	3
12	1545780028345667311380575449*	10	≤ 23
14	**3388031791**	3	3
16	**17476613**	7	7
18	6481049360854613144556866375483521*	5	5
20	76043484008534356379398200621*	7	7
22	**104228508212890623***	3	3
24	25335305376270095455498383578391968*	14	≤ 31
26	**1489106237081787109375***	3	3
28	452785996182923067361779632166688093890621*	7	7

Table 4: Smallest known elements $n \in D(k)$ with largest $L(n)$ for even $k < 30$.

Numbers n which are guaranteed to be the smallest in the corresponding $D(k, L(n))$ are in bold. Sign "$\star$" in the n column showes that corresponding n have been discovered by the author of this paper.

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